MEASURE THEORY AND INTEGRATION – FINAL EXAM

1. Let $E \subset \mathbb{R}$ be a Borel measurable set with finite measure $m(E)$.

   (a) Show that there exists a Borel measurable set $A \subset E$ with $m(A) = m(E)/2$.

   (b) Show that for all $\varepsilon > 0$ there exists an open set B with $E \subset B$ and $m(B) < m(E) + \varepsilon$.

   Solution.

   (a) For $x \geq 0$, define $f(x) = m(E \cap [-x, x])$. f is continuous since

$$\begin{aligned}|f(x+y) - f(x)| &= m(E \cap [-x-y, x+y]) - m(E \cap [-x, x]) \\ &= m(E \cap ([-x-y, -x) \cup (x, x+y])) \\ &= m(E \cap [-x-y, -x)) + m(E \cap (x, x+y]) \\ &= m([-x-y, -x)) + m((x, x+y]) \leq 2y.\end{aligned}$$

   Also note that $f(0) \leq m(\{0\}) = 0$ and the sets $E \cap [-n, n]$ are increasing and their union is equal to E, so

$$\lim_{n\to\infty} f(n) = \lim_{n\to\infty} m(E \cap [-n, n]) = m(E).$$

   By the intermediate value theorem, there exists x such that $f(x) = m(E \cap [-x, x]) = m(E)/2$.

   (b) Fix $\varepsilon > 0$. Recall that, since E is measurable,

$$m(E) = m^*(E) = \inf \left\{ \sum_{i=1}^{\infty} \ell(R_i) : R_1, R_2, \ldots \text{ are rectangles covering } E \right\}.$$

   We can thus choose $R_1, R_2, \ldots$ covering E such that $\sum_{i=1}^{\infty} \ell(R_i) < m(E) + \varepsilon/2$. Now for each i, take an open rectangle R_i' which contains i and such that $m(R_i') = \ell(R_i) + \varepsilon 2^{-i-1}$. We then have $E \subset \cup R_i'$ and

$$\sum_{i=1}^{\infty} m(R_i') \leq \sum_{i=1}^{\infty} (\ell(R_i) + \varepsilon 2^{-i-1}) \leq m(E) + \varepsilon.$$

2. (a) Let f be a non-negative and measurable function defined on a measure space $(\Omega, \mathcal{A}, \mu)$. Show that

$$\mu(\{\omega : f(\omega) > \alpha\}) \leq \frac{1}{\alpha} \int_\Omega f \, d\mu \qquad \forall \alpha > 0.$$

   (b) Show that a measure space $(\Omega, \mathcal{A}, \mu)$ is σ-finite if and only if there exists an integrable function $f : \Omega \to \mathbb{R}$ so that $f(\omega) > 0$ for all ω.

   Solution.

   (a) $f \geq \alpha \cdot \mathbb{1}_{\{\omega : f(\omega) > \alpha\}}$, so

$$\int f \, d\mu \geq \int \alpha \cdot \mathbb{1}_{\{\omega : f(\omega) > \alpha\}} \, d\mu = \alpha \cdot \mu(\{\omega : f(\omega) > \alpha\}).$$

(b) Assume that the space is σ-finite. Then, there exist sets $\Omega_1 \subset \Omega_2 \subset \cdots$ so that $\Omega_n \in \mathcal{A}$, $\mu(\Omega_n) < \infty$ for all n and $\cup_{n=1}^{\infty}\Omega_n = \Omega$. We define

$$f = \sum_{n=1}^{\infty}(2^n \cdot \mu(\Omega_n))^{-1} \cdot \mathbb{1}_{\Omega_n}.$$

We then have $f(\omega) > 0$ for all ω (since every ω belongs to some Ω_n) and

$$\int_\Omega |f| \, d\mu = \int_\Omega f \, d\mu = \sum_{n=1}^{\infty}(2^n \cdot \mu(\Omega_n))^{-1} \cdot \mu(\Omega_n) < \infty,$$

so f is integrable.

Now assume there exists some integrable and strictly positive function f defined on Ω. Let

$$\Omega_n = \{\omega : f(\omega) \geq 1/n\}.$$

The facts that $\Omega_n \subset \Omega_{n+1}$ for each n and $\cup_{n=1}^{\infty}\Omega_n = \Omega$ is evident. Moreover,

$$\frac{1}{n} \cdot \mu(\Omega_n) \leq \int_{\Omega_n} f \, d\mu \leq \int_\Omega f \, d\mu < \infty,$$

so $\mu(\Omega_n) < \infty$ for each n.

3. Assume that $f_n : \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}$, are measurable functions with $f_1 \geq f_2 \geq \cdots \geq 0$ and such that $\int_\mathbb{R} f_n \, \mathrm{d}m \to 0$ (m denotes Lebesgue measure). Prove that $f_n \to 0$ almost everywhere.

Solution. For every x, the sequence $(f_n(x))_{n \geq 1}$ is decreasing and bounded, so we can define $f(x) = \lim_{n\to\infty} f_n(x)$. Since $\int f_n \to 0$, there exists N such that $\int f_n < \infty$ for all $n \geq N$. Then, f_N is an integrable function which dominates $|f|$, $|f_N|$, $|f_{N+1}|$, $|f_{N+2}|$, $\ldots$, so the dominated convergence theorem implies that

$$\lim_{n\to\infty} \int f_n = \lim_{n\to\infty, n\geq N} \int f_n = \int f.$$

On the other hand, we are told that $\int f_n \to 0$, so we have $\int f = 0$. Since $f \geq 0$, we must have $f = 0$ almost everywhere. This shows that $f_n \to 0$ almost everywhere.

4. Let $(\Omega_1, \mathcal{A}_1)$ and $(\Omega_2, \mathcal{A}_2)$ be two measurable spaces.

   (a) Give the definition of the product σ-algebra $\mathcal{A}_1 \otimes \mathcal{A}_2$.

   (b) Show that, for every $A \in \mathcal{A}_1 \otimes \mathcal{A}_2$ and every $\omega_1 \in \Omega_1$, we have $A_{\omega_1} \in \mathcal{A}_2$ (recall that $A_{\omega_1} = \{\omega_2 : (\omega_1, \omega_2) \in A\}$).

Solution.

(a) $\mathcal{A}_1 \otimes \mathcal{A}_2$ is the σ-algebra generated by the collection $\mathcal{G}$ given by

$$\mathcal{G} = \{A_1 \times A_2 : A_1 \in \mathcal{A}_1,\ A_2 \in \mathcal{A}_2\}.$$

(b) Fix $\omega_1 \in \Omega_1$. First assume that $A = A_1 \times A_2$, with $A_1 \in \mathcal{A}_1$, $A_2 \in \mathcal{A}_2$. We then have

$$A_{\omega_1} = \begin{cases} A_2 & \text{if } \omega_1 \in A_1, \\ \varnothing & \text{otherwise.} \end{cases}.$$

Defining $\mathcal{F} = \{A \in \mathcal{A}_1 \otimes \mathcal{A}_2 : A_{\omega_1} \in \mathcal{A}_2\}$, we have proved that $\mathcal{G} \subset \mathcal{F}$. We will now prove that $\mathcal{F}$ is a σ-algebra, so that

$$\mathcal{G} \subset \mathcal{F} \quad \Longrightarrow \quad \mathcal{A}_1 \otimes \mathcal{A}_2 = \sigma(\mathcal{G}) \subset \sigma(\mathcal{F}) = \mathcal{F} \subset \mathcal{A}_1 \otimes \mathcal{A}_2$$

and the proof will be complete. We have:

- $\varnothing \in \mathcal{F}$ is obvious, since $(\varnothing)_{\omega_1} = \varnothing \in \mathcal{A}_2$;
- if $A \in \mathcal{F}$, then $(A^c)_{\omega_1} = (A_{\omega_1})^c \in \mathcal{A}_2$ since $A_{\omega_1} \in \mathcal{A}_2$, so $A^c \in \mathcal{F}$;
- if $A_1, A_2, \ldots \in \mathcal{F}$, then $(\cup A_n)_{\omega_1} = \cup((A_n)_{\omega_1}) \in \mathcal{A}_2$ since each $(A_n)_{\omega_1} \in \mathcal{A}_2$, so $\cup A_n \in \mathcal{F}$.

5. Let $p, q \in (1, \infty)$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$.

(a) Assume $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are sequences of real numbers satisfying

$$\sum_{n=1}^{\infty} |a_n|^p < \infty, \qquad \sum_{n=1}^{\infty} |b_n|^q < \infty.$$

Show that the series $\sum_{n=1}^{\infty}(a_n \cdot b_n)$ is convergent.
Hint. Work on the measure space $(\mathbb{N}, P(\mathbb{N}), \mu)$, where μ is the counting measure. Note that a function $f : \mathbb{N} \to \overline{\mathbb{R}}$ can be associated to a sequence $(f_n)_{n\in\mathbb{N}}$ by setting $f_n = f(n)$. When is a function integrable, and what does integration mean in this space? What is $\mathcal{L}^p$?

(b) Again assume that $(a_n)_{n\in\mathbb{N}}$ satisfies $\sum_{n=1}^{\infty} |a_n|^p < \infty$. Show that

$$\left(\sum_{n=1}^{\infty} |a_n|^p\right)^{1/p} = \sup\left\{\sum_{n=1}^{\infty}(a_n \cdot b_n) : (b_n)_{n\in\mathbb{N}} \text{ is a sequence satisfying } \sum_{n=1}^{\infty} |b_n|^q = 1\right\}.$$

Hint. To show that the left-hand side is less than or equal to the right-hand side, consider

$$b_n^\star = C \cdot \operatorname{sign}(a_n) \cdot (a_n)^{p-1},$$

where $C = (\sum_{n=1}^{\infty} |a_n|^p)^{-1/q}$.

Solution.

(a) As suggested in the hint, we work on the measure space $(\mathbb{N}, P(\mathbb{N}), \mu)$ and for functions $a : \mathbb{N} \to \overline{\mathbb{R}}$ we use the notation a_n rather than $a(n)$ (so that $a = (a_n)_{n\in\mathbb{N}}$). Note that every function is measurable, since we take the power set as the σ-algebra. A function a is integrable if and only if $\sum_{n=1}^{\infty} |a_n| < \infty$ and, in that case, we have $\int_{\mathbb{N}} a_n \, d\mu = \sum_{n=1}^{\infty} a_n$. Furthermore, for $1 \leq p < \infty$, $a \in \mathcal{L}^p$ if and only if $\sum_{n=1}^{\infty} |a_n|^p < \infty$, and then we have

$$\|a\|_p = \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{1/p}.$$

Now, given sequences $a = (a_n)_{n\in\mathbb{N}}$ and $b = (b_n)_{n\in\mathbb{N}}$ so that $\sum_{n=1}^{\infty} |a_n|^p < \infty$ and $\sum_{n=1}^{\infty} |b_n|^q < \infty$, Hölder's inequality gives

$$\sum_{n=1}^{\infty} |a_n \cdot b_n| = \int_{\mathbb{N}} |a \cdot b| \, d\mu \leq \|a\|_p \cdot \|b\|_q = \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{1/p} \cdot \left(\sum_{n=1}^{\infty} |b_n|^q\right)^{1/q} < \infty.$$

Hence, $\sum(a_n \cdot b_n)$ is absolutely convergent, hence it is convergent.

(b) Assume $b = (b_n)_{n\in\mathbb{N}}$ is a sequence satisfying $\sum_{n=1}^{\infty} |b_n|^q = 1$. We then have, as in the previous item

$$\sum_{n=1}^{\infty}(a_n \cdot b_n) \leq \sum_{n=1}^{\infty} |a_n \cdot b_n| \leq \|a\|_p \cdot \|b\|_q = \|a\|_p.$$

This shows that

$$\left(\sum_{n=1}^{\infty} |a_n|^p\right)^{1/p} \geq \sup\left\{\sum_{n=1}^{\infty}(a_n \cdot b_n) : \ (b_n)_{n\in\mathbb{N}} \text{ is a sequence satisfying } \sum_{n=1}^{\infty} |b_n|^q = 1\right\}.$$

In order to prove the reverse inequality, define

$$b_n^\star = C \cdot \operatorname{sign}(a_n) \cdot (a_n)^{p-1}, \ n \in \mathbb{N}, \qquad \text{where } C = \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{-1/q}.$$

Note that

$$\sum_{n=1}^{\infty} |b_n^\star|^q = C^q \sum_{n=1}^{\infty} |a_n|^{(p-1)q} = C^q \sum_{n=1}^{\infty} |a_n|^p = \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{-1+1} = 1.$$

We also have $a_n \cdot b_n^\star = C \cdot |a_n|^p$ for every n, so

$$\sum_{n=1}^{\infty}(a_n \cdot b_n^\star) = C \sum_{n=1}^{\infty} |a_n|^p = \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{1-\frac{1}{q}} = \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{1/p}.$$

This proves that

$$\sup\left\{\sum_{n=1}^{\infty}(a_n \cdot b_n) : \ (b_n)_{n\in\mathbb{N}} \text{ is a sequence satisfying } \sum_{n=1}^{\infty} |b_n|^q = 1\right\} \geq \sum_{n=1}^{\infty}(a_n \cdot b_n^\star) = \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{1/p}.$$